

# Non-vacuum Solutions of Bianchi Type $VI_0$ Universe in $f(R)$ Gravity

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## Abstract

In this paper, we solve the field equations in metric  $f(R)$  gravity for Bianchi type  $VI_0$  spacetime and discuss evolution of the expanding universe. We find two types of non-vacuum solutions by taking isotropic and anisotropic fluids as the source of matter and dark energy. The physical behavior of these solutions is analyzed and compared in the future evolution with the help of some physical and geometrical parameters. It is concluded that in the presence of isotropic fluid, the model has singularity at  $\tilde{t} = 0$  and represents continuously expanding shearing universe currently entering into phantom phase. In anisotropic fluid, the model has no initial singularity and exhibits the uniform accelerating expansion. However, the spacetime does not achieve isotropy as  $t \rightarrow \infty$  in both of these solutions.

**Keywords:**  $f(R)$  theory; Bianchi type  $VI_0$ .

**PACS:** 04.50.Kd

## 1 Introduction

Extended theories of gravity have become a paradigm in bringing suitable cosmological models where a late time accelerated expansion can be achieved.

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The paradigm, consisting of higher order curvature, establishes the  $f(R)$  theory of gravity. This theory has many applications in cosmology and gravity such as inflation, dark energy (DE), local gravity constraints, cosmological perturbations and spherically symmetric solutions in weak and strong gravitational backgrounds. The main motivation of this theory comes from the fact that every unification of fundamental interaction exhibits effective actions containing higher order terms in the curvature invariants. This strategy was adopted in the study of quantum field theory in curved spacetimes [1] as well as in the Lagrangian of string and Kaluza-Klein theories [2].

Higher order terms always give an even number as an order of the field equations. For example,  $R^2$  term produces fourth order field equations [3], term  $R\Box R$  (where  $\Box \equiv \nabla^\mu \nabla_\mu$ ) gives sixth order field equations [4, 5], similarly,  $R\Box^2 R$  yields eighth order field equations [6] and so on. Using conformal transformation, the term with second derivative corresponds to a scalar field. For instance, fourth order gravitational theory corresponds to Einstein theory with one scalar field, sixth order gravity corresponds to Einstein gravity with two scalar fields, *etc.* [4, 7]. In this context, it is easy to show that  $f(R)$  gravity is equivalent to scalar tensor theory as well as to general relativity (GR) with an ideal fluid [8].

In an isotropic and homogeneous spacetimes, the Einstein field equations give rise to the Friedmann equations which are used to describe the evolution of the universe. However, the speedy development in observational cosmology shows that the universe has undergone two phases of cosmic acceleration. The first phase is called inflation [9]-[12] which is believed to have occurred earlier than matter domination [13]-[15]. This accelerating phase is expected to solve horizon problems involved in the big bang cosmology and to explain nearly flat spectrum of temperature anisotropies observed in Cosmic Microwave Background (CMB) radiations [16]. The second accelerating phase has occurred after the matter domination.

The first model of inflation with  $R + \alpha R^2$ , proposed by Starobinsky [9], can lead to an accelerated expansion of the universe due to  $\alpha R^2$  term. The unknown component giving rise to this late-time acceleration is called dark energy (DE) [17]-[21]. The cosmic acceleration has been confirmed by many observational sources such as Supernovae Ia (SNeIa) [22]-[24], Large scale structure (LSS) [25, 26], CMB [27]-[29] and baryon acoustic oscillations (BAO) [30, 31]. The discovery of DE stimulated the idea that cosmic acceleration might originate from some extension of GR. Dark energy models based on  $f(R)$  theory have been extensively studied to explain the late-time

acceleration.

Most of the exact solutions in this theory have been discussed in spherical symmetry. Multamäki and Vilja [32] studied static spherically symmetric vacuum as well as non-vacuum solutions by taking perfect fluid [33]. Caramês and Bezerra [34] found spherically symmetric vacuum solutions in higher dimensions. Capozziello *et al.* [35] analyzed spherically symmetric solutions using Noether symmetry. The structure of relativistic stars in  $f(R)$  theory has been discussed by many authors [36]-[40]. In a recent paper, we have found static spherically symmetric solution in the presence of dust matter and discussed energy-momentum distribution for some well-known  $f(R)$  models [41] using Landau-Lifshitz complex. We have also checked the stability and constant curvature conditions of these models.

In cylindrical symmetry, Azadi *et al.* [42] have studied solutions in Weyl coordinates. They have shown that constant curvature solutions reduce to only one member of the Tian family in GR. Momeni [43] has found that exact constant scalar curvature solution in cylindrical symmetry is applicable to the exterior of a string. Sharif and his collaborators [44, 45] have investigated the exact solutions of plane Bianchi models for both vacuum and non-vacuum cases. Hollestein and Lobo [46] explored exact solutions of the field equations coupled to non-linear electrodynamics.

It has been observed that some large-angle anomalies appear in CMB radiations which violate the statistical isotropy of the universe [47, 48]. Plane Bianchi models (which are homogeneous but not necessarily isotropic) seem to be the most promising explanation of these anomalies. Jaffe *et al.* [49]-[51] investigated that removing a Bianchi component from the WMAP data can account for several large-angle anomalies leaving the universe to be isotropic. Thus the universe may have achieved a slight anisotropic geometry in cosmological models regardless of the inflation. Further, these models can be classified according to whether anisotropy occurs at an early stage or at later times of the universe. The models for the early stage can be modified in a way to end inflation with a slight anisotropic geometry [52]. For the later class, the isotropy of the universe, achieved during inflation, can be distorted by modifying DE [53, 54].

The objective of this paper is to find non-vacuum exact solutions of Bianchi type  $VI_0$  model in the metric  $f(R)$  gravity. The organization of the paper is as follows. In section 2, we present the field equations and some dynamical quantities describing the evolution of the universe. Section 3 provides solution of the field equations in the presence of perfect fluid. In section

4, we find solution for anisotropic fluid. In the last section 5, summary and comparison of both the solutions is given.

## 2 The Model and the Field Equations

The line element for homogeneous and anisotropic Bianchi type  $VI_0$  space-time is given by

$$ds^2 = dt^2 - A^2(t)dx^2 - e^{2\alpha x}B^2(t)dy^2 - e^{-2\alpha x}C^2(t)dz^2, \quad (2.1)$$

where the scale factors  $A$ ,  $B$  and  $C$  are functions of cosmic time  $t$  only and  $\alpha$  is a non-zero constant. There are two formalisms which are applied to derive the field equations in  $f(R)$  gravity. One is the standard metric formalism while another is Palatini formalism. Most of the work in this theory has been done by using the former formalism, where the action with  $S^{(m)}$  as a matter part, is given as follows [55]

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S^{(m)}. \quad (2.1)$$

Its variation with respect to the metric tensor yields the following set of field equations

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \square F(R) = \kappa T_{\mu\nu}, \quad (2.2)$$

where  $F(R) \equiv df(R)/dR$  and  $f(R)$  is a function of the Ricci scalar and describes all kinds of matter including non-relativistic (cold) dark matter. Taking trace of the above equation (with  $\kappa = 1$ ), we obtain  $f(R)$

$$f(R) = \frac{F(R)R + 3\square F(R) - T}{2}. \quad (2.3)$$

The scalar curvature for Bianchi type  $VI_0$  model is given by

$$R = -2 \left[ \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\alpha^2}{A^2} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} \right]. \quad (2.4)$$

The directional Hubble parameters along  $x$ ,  $y$  and  $z$  directions, the average scale factor  $a$ , the deceleration parameter  $q$ , the expansion scalar  $\Theta$ , the

shear scalar  $\sigma$  and the mean Hubble parameter  $H$  are respectively given as

$$H_x = \frac{\dot{A}}{A}, \quad H_y = \frac{\dot{B}}{B}, \quad H_z = \frac{\dot{C}}{C}, \quad a = (ABC)^{1/3}, \quad (2.5)$$

$$q = -\frac{a\ddot{a}}{\dot{a}^2}, \quad \Theta = u^a_{;a} = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}, \quad \sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}, \quad (2.6)$$

$$H = \frac{1}{3}(\ln V)^{\cdot} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right). \quad (2.7)$$

Here  $u^a$  is the four velocity vector and  $\sigma_{ab}$  is the shear tensor. Berman [56, 57] introduced the variation of mean Hubble parameter as follows

$$H = ka^{-n}, \quad (2.8)$$

where  $k > 0$  and  $n \geq 0$ . Inserting this value in Eq.(2.6), we obtain

$$a = c_1 e^{kt}, \quad q = -1 \quad \text{for } n = 0, \quad (2.9)$$

$$a = (nkt + c_2)^{1/n}, \quad q = n - 1 \quad \text{for } n \neq 0, \quad (2.10)$$

where  $c_1$  and  $c_2$  are positive constants of integration. For  $n < 1$ , these equations represent an accelerated expansion of the universe with  $a \rightarrow \infty$  as  $t \rightarrow \infty$  and supports the observations of the Type Ia supernova [22, 23] and WMAP data [27, 28]. To examine whether expansion of the universe is anisotropic or not, we define anisotropic expansion parameter as

$$\Delta = \frac{1}{3} \sum_{i=1}^3 \left( \frac{H_i - H}{H} \right)^2. \quad (2.11)$$

If  $\Delta = 0$ , then the expansion of the universe is isotropic. Further, any anisotropic model of the universe with diagonal energy-momentum tensor approaches to isotropy if  $\Delta \rightarrow 0$ ,  $V \rightarrow +\infty$  and  $\rho > 0$  as  $t \rightarrow +\infty$  [58, 59].

### 3 Solution with Isotropic Fluid

In this section, we take isotropic perfect fluid whose energy-momentum tensor is given by

$$T_{\mu}^{\nu} = \text{diag}[\rho, -p, -p, -p], \quad (3.1)$$

where  $\rho$  is the density and  $p$  is the pressure. The corresponding field equations become

$$\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right)F + \frac{1}{2}f(R) - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\dot{F} = -\rho, \quad (3.2)$$

$$\left(\frac{\ddot{A}}{A} - \frac{2\alpha^2}{A^2} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC}\right)F + \frac{1}{2}f(R) - \ddot{F} - \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\dot{F} = p, \quad (3.3)$$

$$\left(\frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC}\right)F + \frac{1}{2}f(R) - \ddot{F} - \left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\dot{F} = p, \quad (3.4)$$

$$\left(\frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right)F + \frac{1}{2}f(R) - \ddot{F} - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\dot{F} = p, \quad (3.5)$$

$$\alpha\left(\frac{\dot{C}}{C} - \frac{\dot{B}}{B}\right)F = 0. \quad (3.6)$$

The solution of Eq.(3.6) yields

$$C = c_3 B, \quad (3.7)$$

where  $c_3 > 0$  is another constant of integration. Without any loss of generality, we take  $c_3 = 1$  for the sake of simplicity. Using this value of  $C$  in the above equations, we obtain

$$\left(\frac{\ddot{A}}{A} + \frac{2\ddot{B}}{B}\right)F + \frac{1}{2}f(R) - \left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}\right)\dot{F} = -\rho, \quad (3.8)$$

$$\left(\frac{\ddot{A}}{A} - \frac{2\alpha^2}{A^2} + \frac{2\dot{A}\dot{B}}{AB}\right)F + \frac{1}{2}f(R) - \ddot{F} - \frac{2\dot{B}}{B}\dot{F} = p, \quad (3.9)$$

$$\left(\frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2}\right)F + \frac{1}{2}f(R) - \ddot{F} - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\dot{F} = p. \quad (3.10)$$

Subtracting Eq.(3.9) from (3.8), Eq.(3.10) from (3.8) respectively and dividing the resulting equations by  $F$ , we have

$$\frac{-\ddot{B}}{B} - \frac{2\alpha^2}{A^2} + \frac{2\dot{A}\dot{B}}{AB} - \frac{\ddot{F}}{F} + \frac{\dot{A}\dot{F}}{AF} = \frac{\rho + p}{F}, \quad (3.11)$$

$$-\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} - \frac{\ddot{F}}{F} + \frac{\dot{B}\dot{F}}{BF} = \frac{\rho + p}{F}. \quad (3.12)$$

We assume that the scalar expansion is proportional to the shear scalar [60]-[62], i.e.,  $\Theta \propto \sigma$ , which leads to a relation between the metric functions as follows

$$B = A^n, \quad (3.13)$$

where  $n \neq 1$  is a positive constant. Using this relation in Eqs.(3.11) and (3.12), it follows that

$$2n(2-n)\frac{\dot{A}^2}{A^2} - 2n\frac{\ddot{A}}{A} - \frac{2\alpha^2}{A^2} - \frac{\ddot{F}}{F} + \frac{\dot{A}\dot{F}}{AF} = \frac{\rho+p}{F}, \quad (3.14)$$

$$2n\frac{\dot{A}^2}{A^2} - (n+1)\frac{\ddot{A}}{A} - \frac{\ddot{F}}{F} + n\frac{\dot{A}\dot{F}}{AF} = \frac{\rho+p}{F}. \quad (3.15)$$

Subtraction of Eq.(3.15) from (3.14) yields

$$\frac{\ddot{A}}{A} + 2n\frac{\dot{A}^2}{A^2} + \frac{\dot{A}\dot{F}}{AF} - \frac{2\alpha^2}{(1-n)A^2} = 0. \quad (3.16)$$

We solve this equation using power law relation between  $F$  and  $a$  [44],

$$F = la^m, \quad (3.17)$$

where  $l$  is the constant of proportionality and  $m$  is any real number. Substituting the value of  $a$ ,  $F$  turns out to be

$$F = lA^{(\frac{2n+1}{3})m}. \quad (3.18)$$

Inserting this value in Eq.(3.16), it follows that

$$\frac{\ddot{A}}{A} + \frac{6n + (2n+1)m}{3} \frac{\dot{A}^2}{A^2} - \frac{2\alpha^2}{(1-n)A^2} = 0. \quad (3.19)$$

Integrating this equation, we obtain

$$\dot{A} = \sqrt{\frac{6\alpha^2}{(1-n)(m+6n+2mn)} + \frac{c_4}{A^{2(m+6n+2mn)/3}}}, \quad (3.20)$$

where  $c_4$  is another integration constant. With the help of Eqs.(3.7) and (3.13), the line element (2.1) reduces to

$$\begin{aligned} ds^2 &= \left[ \frac{(1-n)(m+6n+2mn)A^{2(m+6n+2mn)/3}}{6\alpha^2 A^{2(m+6n+2mn)/3} + c_4(m+6n+2mn)} \right] dA^2 - A^2 dx^2 \\ &- e^{2\alpha x} A^{2n} dy^2 - c_3^2 A^{2n} e^{-2\alpha x} dz^2. \end{aligned} \quad (3.21)$$

Taking the coordinate transformation ( $A = \tilde{t}$ ), Eq.(3.21) becomes

$$ds^2 = \left[ \frac{(1-n)(m+6n+2mn)\tilde{t}^{2(m+6n+2mn)/3}}{6\alpha^2\tilde{t}^{2(m+6n+2mn)/3} + c_4(m+6n+2mn)} \right] d\tilde{t}^2 - \tilde{t}^2 dx^2 - e^{2\alpha x} \tilde{t}^{2n} dy^2 - c_3^2 \tilde{t}^{2n} e^{-2\alpha x} dz^2. \quad (3.22)$$

Using Eq.(3.18) in Eqs.(3.14) and (3.15), we obtain

$$[2n(2-n) + \frac{2(2n+1)m}{3} - \frac{(2n+1)^2 m^2}{9}] \frac{\dot{A}^2}{A^2} - [2n + \frac{(2n+1)m}{3}] \frac{\ddot{A}}{A} - \frac{2\alpha^2}{A^2} = \frac{\rho + p}{lA^{\frac{(2n+1)m}{3}}}, \quad (3.23)$$

$$[2n + (1-n) \frac{(2n+1)m}{3} - \frac{(2n+1)^2 m^2}{9}] \frac{\dot{A}^2}{A^2} - [(n+1) + \frac{(2n+1)m}{3}] \frac{\ddot{A}}{A} = \frac{\rho + p}{lA^{\frac{(2n+1)m}{3}}}. \quad (3.24)$$

Since the right hand sides of these equations are same with two unknown functions  $\rho$  and  $p$ , thus we get a dependent solution. Using equation of state (EoS)  $p = \omega\rho$ , where  $\omega$  is the EoS parameter and adding Eqs.(3.23) and (3.24), the energy density is found to be

$$\begin{aligned} \rho &= \frac{\tilde{t}^{\frac{(2n+1)m}{3}-2}}{2(1+\omega)} \left[ \left\{ \frac{-6\alpha^2}{(n-1)(m+6n+2mn)} + \frac{c_4}{\tilde{t}^{2(m+6n+2mn)/3}} \right\} \right. \\ &\quad \times \left\{ 6n^2 + m(-2n^2 + \frac{25n}{12} + \frac{5}{3}) + \frac{8}{9}(2n+1)^2 m^2 \right\} \\ &\quad \left. + 4\alpha^2 \left\{ \frac{n+1}{n-1} + \left( \frac{2n+1}{1-n} \right) \frac{m}{3} \right\} \right]. \end{aligned} \quad (3.25)$$

The positive values of  $\omega$  characterize different kinds of fluids while in order to explain DE of the current universe with the help of Bianchi models, one can use  $\omega$  with negative values. When  $w = -1$ , the universe passes through  $\Lambda$ CDM epoch. If  $w < -1$ , then we live in the phantom-dominated universe and for  $w > -1$ , the quintessence dark era occurs. Here we analyze the behavior of energy density in phantom and quintessence regions for different positive values of  $m$  as shown in Figure 1 while  $\omega = -1$  yields singular expression. It can be observed that our solution favors only the phantom region because  $\rho$  is negative in the other region which is not feasible. Thus



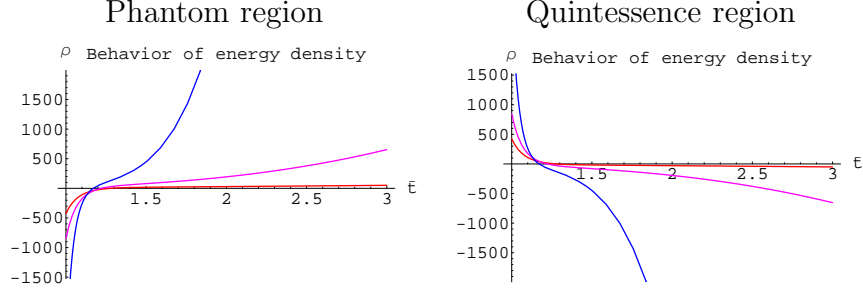


Figure 1: Behavior of energy density versus cosmic time. The three colored lines are sketched for  $m = 2$ ,  $m = 3$  and  $m = 4$  respectively.

the energy density will increase and the universe enters into the phantom phase. It is worthwhile to mention here that observational analysis of a recent supernova strongly supports  $w < -1$  being EoS parameter for phantom DE [63]-[65].

For this model, the physical quantities will become

$$\begin{aligned} H &= \frac{(2n+1)}{3\tilde{t}^{m+6n+2mn+1}} \sqrt{-2\alpha^2 \tilde{t}^{2(m+6n+2mn)} + c_4(n-1)(m+6n+2mn)}, \\ V &= \tilde{t}^{2n+1}. \end{aligned} \quad (3.26)$$

The expansion scalar  $\Theta$  and the shear scalar  $\sigma$  become

$$\Theta = \frac{(2n+1)(-6\alpha^2 \tilde{t}^{2(m+6n+2mn)/3} + c_4(n-1)(m+6n+2mn))^{1/2}}{\tilde{t}^{\frac{m+6n+2mn}{3}+1}}, \quad (3.27)$$

$$\sigma = \frac{(1-n)(-6\alpha^2 \tilde{t}^{2(m+6n+2mn)} + c_4(n-1)(m+6n+2mn))^{1/2}}{\sqrt{3} \tilde{t}^{\frac{m+6n+2mn}{3}+1}}. \quad (3.28)$$

We see that the scale factors and volume of the universe are zero at initial epoch which shows that the model has point type singularity while they continue to increase with time. The Hubble parameter and the expansion scalar indicate that expansion rate was rapid at initial times of the big bang but it slows down with the passage of time and tends to zero as  $\tilde{t} \rightarrow \infty$ . The ratio  $\frac{\sigma}{\Theta}$  indicates that the universe does not achieve isotropy and hence model represents continuously expanding, shearing universe from the start of the big bang.

The scalar curvature and  $f(R)$  function turn out to be

$$R = \frac{2\alpha^2(5n+1)}{\tilde{t}^2(n-1)} + 2(5n^2 + 6n + (2n+1)^2m) \\ \times \frac{-2\alpha^2\tilde{t}^{2(m+6n+2mn)} + c_4(n-1)(m+6n+2mn)}{(n-1)(m+6n+2mn)\tilde{t}^{2(m+6n+2mn+1)}}, \quad (3.29)$$

$$f(R) = l\tilde{t}^{(2n+1)/3}R + 3(n+1) \\ \times \left[ \frac{-2\alpha^2}{(n-1)(m+6n+2mn)\tilde{t}^2} + \frac{c_4}{\tilde{t}^{2(m+6n+2mn+1)}} \right]^{1/2}. \quad (3.30)$$

We know that if  $f(R)$  is replaced by  $R + \Lambda$ , then  $f(R)$  theory corresponds to GR, where  $\Lambda$  is interpreted as the energy density of the vacuum [66]–[69] causing expansion in the universe. Thus  $f(R)$  may be used to explain the present cosmological expansion. We note that the above function shows initial time singularity and continuously expanding to infinity.

## 4 Solution with Anisotropic Fluid

Here, we obtain solution of the field equations for anisotropic fluid

$$T_\mu^\nu = \text{diag}[\rho, -p_x, -p_y, -p_z] = \text{diag}[1, -\omega_x, -\omega_y, -\omega_z]\rho, \quad (4.1)$$

where  $p_x$ ,  $p_y$  and  $p_z$  are pressures and  $\omega_x$ ,  $\omega_y$  and  $\omega_z$  are directional EoS parameters on  $x$ ,  $y$  and  $z$  axes respectively. We take

$$\omega_x = \omega + \delta, \quad \omega_y = \omega \quad \text{and} \quad \omega_z = \omega + \gamma,$$

where  $\omega$  is the deviation-free EoS parameter and  $\delta$  and  $\gamma$  (called skewness parameters) are deviations from  $\omega$  on  $x$  and  $z$  axes respectively. The energy-momentum tensor takes the form

$$T_\mu^\nu = \text{diag}[1, -(\omega + \delta), -\omega, -(\omega + \gamma)]\rho. \quad (4.2)$$

Notice that  $\omega$ ,  $\delta$  and  $\gamma$  are functions of cosmic time  $t$ . The field equations will become

$$\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right)F + \frac{1}{2}f(R) - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\dot{F} = -\rho, \quad (4.3)$$

$$\left(\frac{\ddot{A}}{A} - \frac{2\alpha^2}{A^2} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC}\right)F + \frac{1}{2}f(R) - \ddot{F} - \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\dot{F} = (\omega + \delta)\rho, \quad (4.4)$$

$$\left(\frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC}\right)F + \frac{1}{2}f(R) - \ddot{F} - \left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)\dot{F} = \omega\rho, \quad (4.5)$$

$$\left(\frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right)F + \frac{1}{2}f(R) - \ddot{F} - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\dot{F} = (\omega + \gamma)\rho, \quad (4.6)$$

$$\alpha\left(\frac{\dot{C}}{C} - \frac{\dot{B}}{B}\right)F = 0. \quad (4.7)$$

Using solution of Eq.(4.7) in the above system and then subtracting Eq.(4.5) from (4.6), we obtain  $\gamma = 0$ . This indicates that the directional EoS parameters  $\omega_y$ ,  $\omega_z$  along  $y$  and  $z$  axes become equal and hence also pressure. Consequently, the field equations turn out to be

$$\left(\frac{\ddot{A}}{A} + \frac{2\ddot{B}}{B}\right)F + \frac{1}{2}f(R) - \left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}\right)\dot{F} = -\rho, \quad (4.8)$$

$$\left(\frac{\ddot{A}}{A} - \frac{2\alpha^2}{A^2} + \frac{2\dot{A}\dot{B}}{AB}\right)F + \frac{1}{2}f(R) - \ddot{F} - \frac{2\dot{B}}{B}\dot{F} = (\omega + \delta)\rho, \quad (4.9)$$

$$\left(\frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2}\right)F + \frac{1}{2}f(R) - \ddot{F} - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\dot{F} = \omega\rho. \quad (4.10)$$

Subtracting Eq.(4.9) from (4.10) and integrating the resulting equation, we have

$$H_x - H_y = \frac{c_5}{VF} + \frac{1}{VF} \int \left(\frac{\alpha^2 F}{A^2} + \delta\rho\right) V dt, \quad (4.11)$$

where  $c_5$  is another positive constant of integration. Using Eqs.(2.11) and (4.11), it follows that

$$\Delta = \frac{2}{9H^2} [c_5 + \int \left(\frac{\alpha^2 F}{A^2} + \delta\rho\right) V dt]^2 V^{-2} F^{-2}. \quad (4.12)$$

If we take  $\delta = 0$  and  $F(R) = 1$ , the anisotropy parameter of expansion reduces to GR for an isotropic fluid. To avoid the integral term in Eq.(4.12), we can choose  $\delta$  such that

$$\delta = -\frac{2\alpha^2 F}{\rho A^2}. \quad (4.13)$$

The energy-momentum tensor, the anisotropy parameter and Eq.(4.11) will take the form

$$T_\mu^\nu = \text{diag}[1, -\omega + \frac{2\alpha^2 F}{\rho A^2}, -\omega, -\omega]\rho, \quad (4.14)$$

$$\Delta = \frac{2}{9} \frac{c_5^2}{H^2} V^{-2} F^{-2}, \quad (4.15)$$

$$H_x - H_y = \frac{c_5}{VF}. \quad (4.16)$$

Using Eq.(4.13) in (4.9), we obtain the following set of field equation

$$(\frac{\ddot{A}}{A} + \frac{2\ddot{B}}{B})F + \frac{1}{2}f(R) - (\frac{\dot{A}}{A} + \frac{2\dot{B}}{B})\dot{F} = -\rho, \quad (4.17)$$

$$(\frac{\ddot{A}}{A} + \frac{2\dot{A}\dot{B}}{AB})F + \frac{1}{2}f(R) - \ddot{F} - \frac{2\dot{B}}{B}\dot{F} = \omega\rho, \quad (4.18)$$

$$(\frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2})F + \frac{1}{2}f(R) - \ddot{F} - (\frac{2\dot{A}}{A} + \frac{2\dot{B}}{B})\dot{F} = \omega\rho. \quad (4.19)$$

We can solve this system of equations by using exponential and power law volumetric expansions corresponding to Eqs.(2.9) and (2.10) respectively. The exponential expansion (also known as de Sitter expansion) has been discussed in detail [70]. Using Eq.(2.9), the volume of the universe and  $F$  can be written as follows

$$V = c_1^3 e^{3kt}, \quad (4.20)$$

$$F = l c_1^m e^{mkt}. \quad (4.21)$$

Inserting Eqs.(4.20) and (4.21) in (4.16), we have

$$H_x - H_y = \frac{c_5 e^{-(3+m)kt}}{l c_1^{3+m}}. \quad (4.22)$$

Solving the system of equations (4.17)-(4.19) along with Eq.(4.22), we obtain the scale factors as follows

$$A = c_6 e^{kt - \frac{2}{3} \frac{c_5}{l(3+m)kc_1^{3+m}} e^{-(3+m)kt}}, \quad (4.23)$$

$$B = C = c_7 e^{kt + \frac{1}{3} \frac{c_5}{l(3+m)kc_1^{3+m}} e^{-(3+m)kt}}, \quad (4.24)$$

where  $c_6$  and  $c_7$  are positive constants of integration. When  $m > -3$ ,

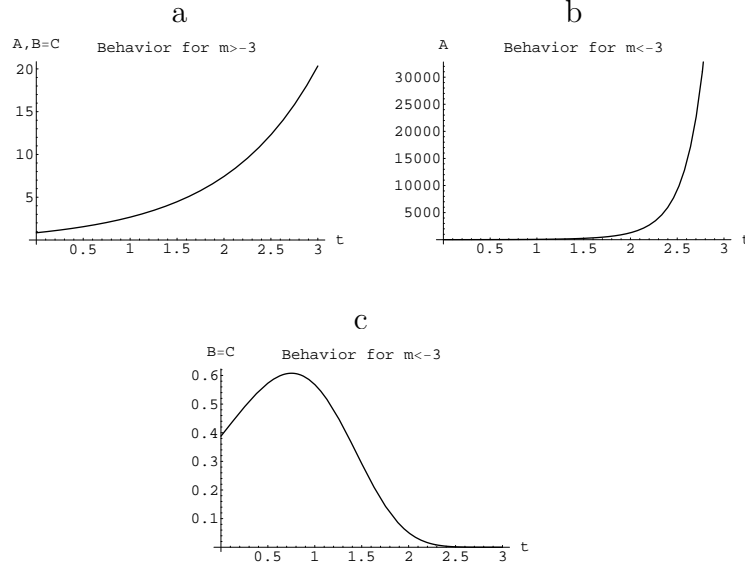


Figure 2: Behavior of scale factors versus time

the scale factors admit constant values at initial time afterwards they start increasing with cosmic time without any type of initial singularity and finally diverges to  $\infty$  as  $t \rightarrow \infty$  (Figure **2a**). This shows that at the initial epoch, the universe starts with some constant volume and expands exponentially approaching to infinite volume. However, for  $m < -3$ , the scale factor  $A$  increases rapidly with time (Figure **2b**) while  $B$  first increases for a finite value of time and then decreases approaching to zero at later times (Figure **2c**). Moreover, the expansion scalar is constant showing that the universe is expanding uniformly from  $t = 0$  to  $t = \infty$ .

The directional and mean Hubble parameters become

$$\begin{aligned} H_x &= k + \frac{2c_5}{3lc_1^{3+m}} e^{-(3+m)kt}, & H_y = H_z &= k - \frac{c_5}{3lc_1^{3+m}} e^{-(3+m)kt}, \\ H &= k. \end{aligned} \quad (4.25)$$

The anisotropy parameters turns out to be

$$\Delta = \frac{2}{9} \frac{c_5^2}{k^2 l^2 c_1^{2(3+m)k}} e^{-2(m+3)kt}. \quad (4.26)$$

We see that the mean Hubble parameter is constant whereas others are time dependent. As time approaches from zero to infinity (for  $m > -3$ ), the directional Hubble parameters are reduced to the mean value. For  $m < -3$ , parameter along  $x$ -axis will increase from the mean Hubble parameter by twice a constant factor (in the coefficient of exponential function) whereas parameters along  $y$  and  $z$ -axes decrease by the same factor. Notice that at  $t = 0$ , the anisotropy parameter measures a constant value while it vanishes for  $m > -3$  at infinite time of the universe. This indicates that the universe expands isotropically at later times without taking any effect from anisotropy of the fluid. However, for  $m < -3$ , the anisotropy in the expansion will increase with time.

Using values of the scale factors in Eq.(4.8), we obtain the following energy density of anisotropic dark fluid

$$\rho = 3l(m-1)k^2c_1^m e^{mkt} - \frac{1}{2}f(R) + \frac{2c_5^2}{3lc_1^{6+m}}e^{-(6+m)kt} \quad (4.27)$$

which involves  $f(R)$  function. Thus if we interpret this function as a simplest model corresponding to  $\Lambda$ CDM, we may observe that at the initial epoch, the energy density is proportional to a constant vacuum energy density  $\Lambda$  only. However, with the passing time, it acquires dynamical terms describing matter densities of fluids filling the universe. These terms will expand to infinity in the future evolution of the universe. This indicates that the energy density is increasing in the future due to creation of some phantom DE and decay of some other components of energy in the universe, like, cold dark matter. Consequently, expansion of the universe is accelerated forever.

Inserting the values of  $A$ ,  $F$ ,  $\rho$  in Eq.(4.13), we obtain skewness parameter  $\delta$

$$\delta = \frac{-12\alpha^2 l^2 c_1^{2(3+m)} e^{mkt-2kt+\frac{4}{3}\frac{c_5}{l(3+m)kc_1^{3+m}}e^{-(3+m)kt}}}{c_6^2[18l^2k^2(m-1)c_1^{4+m}e^{mkt} - 4c_5^2c_1^{-(2+m)}e^{-(6+m)kt} - 3lc_1^4f(R)]}. \quad (4.28)$$

Using this value of  $\delta$  alongwith scale factors and  $\rho$  in Eqs.(4.17)-(4.19) and solving them simultaneously, the anisotropic EoS parameter  $\omega$  will become

$$\omega = \frac{6(1-m)c_1^{3+m}l^2k^2e^{mkt} - 8c_5lmke^{-3kt} + 6lc_1^3f(R)}{18l^2k^2(m-1)c_1^{4+m}e^{mkt} - 4c_5^2c_1^{-(2+m)}e^{-(6+m)kt} - 3lc_1^4f(R)}. \quad (4.29)$$

Since both the parameters contain  $\rho$  and hence  $f(R)$  function, it may not be possible to discuss their behavior independently in the future evolution. The

scalar curvature for exponential solution becomes

$$R = -10k^2 - \frac{2}{3} \frac{c_5^2}{l^2 c_1^{2(3+m)}} e^{-2(3+m)kt} + \frac{2\alpha^2}{c_6^2} e^{-2kt + \frac{4}{3} \frac{c_5}{l(3+m)kc_1^{3+m}} e^{-(3+m)kt}}. \quad (4.30)$$

This procedure can be extended to the solution of the field equations for power-law volumetric expansion.

## 5 Outlook

This paper is devoted to study the exact solutions of the Bianchi type  $VI_0$  universe in the metric  $f(R)$  gravity and to discuss the recent cosmic acceleration. Two types of non-vacuum solutions are found corresponding to isotropic and anisotropic fluids respectively. The physical behavior of these solutions is discussed at early and late times of the universe, which can be summarized as follows:

- The scale factors in the isotropic fluid case are zero at  $\tilde{t} = 0$  which shows that the spacetime exhibits point type singularity and continues to expand till  $\tilde{t} \rightarrow \infty$ . In anisotropic case, the scale factors admit constant value at initial epoch and then start increasing with cosmic time approaching to very large values as  $t \rightarrow \infty$ . Thus, the spacetime does not show any type of initial singularity in this case.
- The model starts with physical parameters such that all being infinite at early times and approaches to zero at later times in the perfect fluid case. In the anisotropic case, the expansion scalar and the mean Hubble parameter are constant indicating homogenous expansion of the universe. The volume of the universe will increase with time due to expansion for both the solutions.
- The matter density and pressure of the fluid are related by the EoS, where EoS parameter is used to characterize the DE into different expansion histories. It is found that in isotropic solution, the DE has large negative pressure with  $\omega < -1$  which shows that the universe passes through phantom region. It is mentioned here that our conclusions support the observational evidence of a recent supernova data [63]-[65]. However, the energy density of the anisotropic fluids involves

$f(R)$  function whose interpretation is given in the context of  $\Lambda$ CDM. In this case, the EoS parameter turns out to be time dependent.

- In the anisotropic solution, the deviation from the isotropy along  $z$ -axis is zero which shows that pressure of DE along  $y$  and  $z$  axes are same. The skewness parameter along  $x$ -axis does not vanish even at the later times. However, for  $m > -3$ , the anisotropy parameter of the expansion becomes zero at infinite times showing that the universe expands isotropically regardless of the anisotropy in the fluid. It is mentioned here that in GR, the anisotropic fluid as well as the model attain isotropy in the future evolution of the universe [58, 71] but in  $f(R)$  theory, the model and the fluid remain anisotropic. Further, in the isotropic fluid case, the universe does not achieve isotropy and the model expands continuously with non-zero shear scalar. It is remarked here that the above mentioned aspect of the model is compatible with existing results of GR [60, 62].

### Acknowledgment

We would like to thank the Higher Education Commission, Islamabad, Pakistan for its financial support through the *Indigenous Ph.D. 5000 Fellowship Program Batch-III*.

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